A new representation of rotational flow fields satisfying Euler's equation of an ideal compressible fluid

Tsutomu Kambe ‡

Higashi-yama 2-11-3, Meguro-ku, Tokyo, Japan 153-0043

E-mail: kambe@ruby.dti.ne.jp

Abstract. A new representation of solution to Euler's equation of motion is presented by a system of expressions for compressible rotational flows of an ideal fluid. This is regarded as generalization of the Bernoulli's theorem to compressible rotational flows. Present expressions are derived from the variational principle. The action functional for the principle consists of main terms of total kinetic, potential and internal energies, together with three additional terms yielding the equations of continuity, entropy and the third term which provides rotational component of velocity field. The last term has a form of scalar product satisfying gauge symmetry with respect to both translation and rotation. This is a generalization of the Clebsch transformation from a physical point of view. It is verified that the system of new expressions in fact satisfies Euler's equation of motion.

Keywords: Euler-equation; Rotational-flow; Compressible-fluid; Bernoulli-equation; Clebsch-transformation

‡ Formerly Professor at Department of Physics, University of Tokyo.

1. Introduction

One of the well-known integrals of Euler's equation of motion is the Bernoulli equation. Historically it had been given in 1738 by Daniel Bernoulli, before Leonhard Euler (1755) proposed his seminal equation of motion in the form of a partial differential equation. The Bernoulli equation are known with various forms: one of the forms holds along each streamline in steady flows of an inviscid fluid, another one is valid at all points in unsteady irrotational inviscid flow, and others.

For an inviscid fluid, Euler's equation of motion is expressed as

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\,\nabla p - \nabla\Psi_e, \tag{1}$$

where $\boldsymbol{v}$, p and ρ are the fluid velocity, pressure and density respectively, Ψ_e the potential of an external force, $\partial_t \equiv \partial/\partial t$ with t the time, and $\nabla = (\partial_i)$ with $\boldsymbol{x} = (x^1, x^2, x^3)$ a point in the three-dimensional cartesian space (where $\partial_i \equiv \partial/\partial x^i$). The above equation can be rewritten as

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla(h + \Psi_e) + T\,\nabla s, \tag{2}$$

To obtain this, the term $\rho^{-1}\nabla p$ is eliminated by using the first of the following thermodynamic relations:

$$\rho^{-1}\,\mathrm{d}p = \mathrm{d}h - T\,\mathrm{d}s, \qquad \mathrm{d}\epsilon = T\,\mathrm{d}s - p\;\mathrm{d}\rho^{-1}, \tag{3}$$

where ϵ, s and h are the *specific* internal energy, entropy and enthalpy (*i.e.* per unit mass) with $h = \epsilon + p/\rho$. A uniform fluid of a single component is characterized by two thermodynamic state variables such as $\epsilon = \epsilon(\rho, s)$, and $h = h(\rho, s)$ with ρ and s two independent variables. Then, the temperature and pressure are defined by $T = (\partial\epsilon/\partial s)_\rho$ and $p = -(\partial\epsilon/\partial\rho^{-1})_s$.

Furthermore, using the following vector *identity*,

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla(\tfrac{1}{2}\,v^2) - \boldsymbol{v}\times\boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \nabla\times\boldsymbol{v}, \tag{4}$$

($\boldsymbol{\omega}$ is the vorticity and $v^2 = |\boldsymbol{v}|^2$), the Euler equation (2) can be rewritten as

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla(h + \tfrac{1}{2}\,v^2 + \Psi_e) + T\,\nabla s, \tag{5}$$

For an *isentropic* flow (in which s is uniform), this reduces to

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla(\tfrac{1}{2}\,v^2 + h + \Psi_e\,). \tag{6}$$

(a) In a steady flow of an inviscid isentropic fluid (in which $\partial_t\boldsymbol{v} = 0$), the scalar product of (6) with $\boldsymbol{v}$ suggests the following relation, (a) $\frac{1}{2}\,v^2 + h + \Psi_e = const$, along each stream line, since the left hand side vanishes by the scalar product.

(b) In two-dimensional flows, we have $\boldsymbol{\omega}\perp\boldsymbol{v}$. In addition if it is steady and the entropy s is non-uniform, the equation (5) reduces to $\nabla(h + \frac{1}{2}\,v^2 + \Psi_e) = T\,\nabla s + \boldsymbol{v}\times\boldsymbol{\omega}$. Taking scalar product with a unit normal vector defined by $\boldsymbol{n} = (\boldsymbol{v}\times\boldsymbol{\omega})/|\boldsymbol{v}||\boldsymbol{\omega}|$, we obtain a Bernoulli equation for variable entropy s and non-zero $\boldsymbol{\omega}$:

$$\frac{\mathrm{d}}{\mathrm{d}n}(\tfrac{1}{2}\,v^2 + h + \Psi_e) = T\,\frac{\mathrm{d}s}{\mathrm{d}n} + |\boldsymbol{v}|\,|\boldsymbol{\omega}|, \qquad \text{along each stream line,} \tag{7}$$

where $\mathrm{d}/\mathrm{d}n = \boldsymbol{n}\cdot\nabla$. This is referred to as *Crocco's theorem* (Liepmann & Roshko 1957; Shapiro 1969).

(c) For the case of irrotational flow in which $\boldsymbol{v}$ can be expressed as $\nabla\Phi$ with a velocity potential Φ, the equation (6) reduces to the following Bernoulli equation valid at any point:

$$\tfrac{1}{2}\,v^2 + h + \Psi_e + \partial_t\Phi = const\ . \tag{8}$$

One may ask a question whether we can generalize the Bernoulli equations to rotational flows of a compressible ideal fluid.

In the section 2, a new representation (or a kind of transformation) is presented, which solves Euler's equation of motion. This is regarded as an extension of the Bernoulli's representation (8) to general *unsteady, rotational* flows of an ideal *compressible* fluid. This generalization is carried out by the variational principle, described in §4.

Earlier studies of the variational formulation of fluid flows were made by Eckart (1938, 1960) and Herivel (1955). But the earliest may be the work of Clebsch (1859). Their variations are carried out in two ways: *i.e.* a Lagrangian approach and an Eulerian approach. In both approaches, the equation of continuity and the condition of entropy are taken into account as constraint conditions by means of Lagrange multipliers.

In the Lagrangian approach, the Euler-Lagrange equation of the variation results in an equation equivalent to Euler's equation of motion. In the Eulerian representation, a common feature of Herivel (1955) and Eckart (1960) is that both arrive at the Clebsch solution which is summarized in Appendix A. However, it was remarked by Bretherton (1970) that the Clebsch representation has only local validity in the neighborhood of a chosen point if vortex lines are knotted or linked. In the Eulerian description, the action principle for isentropic flows results in potential flows (Kambe 2003a, 2007b). This is a long-standing problem (Serrin 1959; Lin 1963; Seliger & Whitham 1968; Bretherton 1970; Salmon 1988). Lin (1963) and Salmon (1988) tried to resolve this difficulty by introducing a *constraint* as a side condition, imposing invariance of Lagrangian particle labels along particle trajectories.

In the present formulation, the velocity is represented by scalar potentials and vector potentials of frozen property, in particular the latter vector potentials is convected with the fluid flow under stretching effect. It is verified in §3 that the system of new expressions *in fact* satisfies Euler's equation of motion. In §4, the Lagrangian for the principle of least action is defined by main terms of total kinetic energy, internal energy (with negative sign) and potential energy (with negative sign), together with three terms yielding the equations of continuity and entropy and the third which yields a *new rotational component* of velocity field (Kambe 2008; 2010). Thus, the velocity $\boldsymbol{v}$ is represented by two scalar potentials ϕ and ψ and in addition by two solenoidal fields of a tangent vector $\bar{\boldsymbol{Z}}$ and a cotangent vector $\boldsymbol{U}$. Present formulation has some difference in the definition of the third term from the previous one given by Kambe (2008; 2010). This slight change has enabled present *explicit* representation in terms of potentials, which is explained in §5 (*ii*). The present representation is compared with that of Clebsch in §5 (*iii*). Characteristic features of the present formulation are discussed in §6 from physical and mathematical point of view.

Once the velocity field $\boldsymbol{v}(t,\boldsymbol{x})$ is given, one can define the convective derivative D_t (*i.e.* the material derivative) by

$$\mathrm{D}_t \equiv \partial_t + \boldsymbol{v}\cdot\nabla\,. \tag{9}$$

In the mathematics of differential forms, Lie-derivatives of a tangent vector $\boldsymbol{Z}(t,\boldsymbol{x}) = (Z^i)$ and a cotangent vector $\boldsymbol{U}(t,\boldsymbol{x}) = (U_i)$ (a *one-form*) are defined by

$$(\mathcal{L}_t[\boldsymbol{Z}])^i \equiv \partial_t Z^i + v^k\partial_k Z^i - Z^k\partial_k v^i, \qquad (\mathcal{L}_t^*[\boldsymbol{U}])_i \equiv \partial_t U_i + v^k\partial_k U_i + U_k\partial_i v^k. \tag{10}$$

respectively. A scalar product of a tangent vector $\boldsymbol{T} = (T^i)$ and a cotangent vector $\boldsymbol{C} = (C_i)$ is defined by the symbol $\langle \boldsymbol{C},\,\boldsymbol{T}\rangle \equiv C_i\,T^i$.§

2. General representation of a family of solutions

The following set of equations is a *new* general representation of a family of solutions to the Euler equation (2) for an ideal compressible fluid:

$$\boldsymbol{v} = \nabla\phi + s\,\nabla\psi + \boldsymbol{w}, \tag{11}$$

$$\boldsymbol{w} = \boldsymbol{Z}\times(\nabla\times\boldsymbol{U}), \tag{12}$$

$$\tfrac{1}{2}\,v^2 + h + \Psi_e + \partial_t\phi + s\,\partial_t\psi - \boldsymbol{v}\cdot\boldsymbol{w} = const\ , \tag{13}$$

$$\partial_t\rho + \nabla\cdot(\rho\,\boldsymbol{v}) = 0. \tag{14}$$

$$\mathrm{D}_t s = 0, \tag{15}$$

$$\mathrm{D}_t\psi + (\partial\epsilon/\partial s)_\rho = 0, \tag{16}$$

$$\mathcal{L}_t^*[\boldsymbol{U}] = 0, \tag{17}$$

$$\partial_i U_i = 0, \tag{18}$$

$$\mathcal{L}_t[\boldsymbol{Z}] = 0, \tag{19}$$

$$\partial_k \bar{Z}^k = 0 \qquad (\text{where}\quad \bar{\boldsymbol{Z}} = \rho\boldsymbol{Z}), \tag{20}$$

where $(\partial\epsilon/\partial s)_\rho = T$ (temperature). The third term $\boldsymbol{w}$ in the expression of $\boldsymbol{v}$ is new as well as the last term of (13), where

$$\boldsymbol{w} = (w_i) = \boldsymbol{Z}\times(\nabla\times\boldsymbol{U}) = Z^k\,\nabla U_k - (Z^k\partial_k)\boldsymbol{U}, \tag{21}$$

$$w_i \;= Z^k\,U_{ik}, \qquad U_{ik} = \partial_i U_k - \partial_k U_i. \tag{22}$$

The expression (13) is a generalized form of the Bernoulli equation (8). From (11) for $\boldsymbol{v}$, one obtains the vorticity $\boldsymbol{\omega}$, given as $\boldsymbol{\omega} = \nabla s\times\nabla\psi + \nabla\times\boldsymbol{w}$. The first term vanishes in the case of uniform entropy where $\nabla s = 0$. Significance of the second term lies in

§ In the euclidian space of metric tensor δ_{ij}, a cotangent-vector version of the tangent vector T^j is $T_i = \delta_{ij}T^j = T^i$. In regard to the Lie derivative $\mathcal{L}_t$, when applied to a scalar function $f(t,\boldsymbol{x})$, the Lie derivative $\mathcal{L}_t$ reduces to D_t. In fact, we have $\mathcal{L}_t[f] = \mathrm{D}_t f = \partial_t f + \boldsymbol{v}\cdot\nabla f$, and $\mathcal{L}_t\langle \boldsymbol{C},\,\boldsymbol{T}\rangle = \langle \mathcal{L}_t^*[\boldsymbol{C}],\,\boldsymbol{T}\rangle + \langle \boldsymbol{C},\,\mathcal{L}_t[\boldsymbol{T}]\rangle = \mathrm{D}_t\langle \boldsymbol{C},\,\boldsymbol{T}\rangle$. Note that ∇f satisfies $\mathcal{L}_t^*[\nabla f] = 0$ if $\mathrm{D}_t f = 0$. In fact, we have $\mathrm{D}_t\nabla f = \nabla(\mathrm{D}_t f) - (\partial_k f)\nabla v^k = -(\partial_k f)\nabla v^k$.

the fact that the flow remains rotational even in the isentropic case. This point will be investigated later in §4.2.

The equations (14) and (15) express conservation of mass and specific entropy respectively. In addition, introducing $\bar{\boldsymbol{Z}}$ defined by $\rho \boldsymbol{Z}$, we can show the following equation for $\bar{\boldsymbol{Z}}$ $(\partial_k \bar{Z}^k = 0)$:

$$\partial_t \bar{\boldsymbol{Z}} + \nabla \times (\bar{\boldsymbol{Z}} \times \boldsymbol{v}) = \mathcal{L}_t[\bar{\boldsymbol{Z}}] + \bar{\boldsymbol{Z}}\, \partial_k v^k = \rho\, \mathcal{L}_t[\boldsymbol{Z}] = 0, \tag{23}$$

by (19), since $\mathcal{L}_t[\bar{\boldsymbol{Z}}] = \mathcal{L}_t[\rho \boldsymbol{Z}] = \rho \mathcal{L}_t[\boldsymbol{Z}] + \boldsymbol{Z}\, \mathrm{D}_t \rho$ and $\mathrm{D}_t \rho + \rho \partial_k v^k = 0$ by (14). The above equation states that the vector field $\bar{\boldsymbol{Z}}$ is carried along with the flow, namely frozen to the flow field $\boldsymbol{v}(t, \boldsymbol{x})$. Later, it will be shown that the vorticity $\boldsymbol{\omega}$ satisfies the same equation. The effect of frozen field is essential in the dynamics of the vorticity.

Similarly, defining $\boldsymbol{\Omega}$ by $\nabla \times \boldsymbol{U}$, it is straightforward to show that $\boldsymbol{\Omega} = \nabla \times \boldsymbol{U}$ satisfies the same equation as that of $\bar{\boldsymbol{Z}}$. Namely, owing to $\nabla \cdot \boldsymbol{\Omega} = 0$, we have

$$\partial_t \boldsymbol{\Omega} + \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{v}) = 0. \tag{24}$$

3. Euler's equation of motion is satisfied

The flow field given by (11) $\sim$ (20) represents that of compressible and rotational flows in general, and satisfies the Euler equation. The proof is carried out as follows.

Applying the derivative D_t to $\boldsymbol{v} = \nabla \phi + s\, \nabla \psi + \boldsymbol{w}$ of (11), we have

$$\mathrm{D}_t[\boldsymbol{v}] = \mathrm{D}_t \nabla\, \phi + \mathrm{D}_t (s \nabla \psi) + \mathrm{D}_t \boldsymbol{w}, \tag{25}$$

The first term can be rewritten as

$$\mathrm{D}_t (\nabla\, \phi) = \nabla (\mathrm{D}_t \phi) - \partial_k \phi\, \nabla v^k. \tag{26}$$

Using two equations of (15), the second term of (25) is

$$\mathrm{D}_t (s\, \nabla \psi) = s\, \nabla (\mathrm{D}_t \psi) - s\, \partial_k \psi\, \nabla v^k \tag{27}$$

By using (22) where $w_i = Z^k U_{ik}$, the third term is

$$\mathrm{D}_t w_i = \mathrm{D}_t (Z^k)\, U_{ik} + Z^k\, \mathrm{D}_t (U_{ik}). \tag{28}$$

Since $\mathcal{L}_t[\boldsymbol{Z}] = 0$, we have $\mathrm{D}_t(Z^k) = Z^l\, \partial_l v^k$. In regard to the term $\mathrm{D}_t(U_{ik})$ where $U_{ik} = \partial_i U_k - \partial_k U_i$, we have

$$\mathrm{D}_t U_{ik} = \partial_i (\mathrm{D}_t U_k) - \partial_k (\mathrm{D}_t U_i) - (\partial_i v^l)\, \partial_l U_k + (\partial_k v^l)\, \partial_l U_i\, .$$

Substituting these two into (28),

$$\begin{aligned} \mathrm{D}_t w_i = {} & Z^l\, \partial_l v^k\, (\partial_i U_k - \partial_k U_i) \\ & + Z^k \Big(\partial_i (\mathrm{D}_t U_k) - \partial_k (\mathrm{D}_t U_i) - (\partial_i v^l)\, \partial_l U_k + (\partial_k v^l)\, \partial_l U_i \Big). \end{aligned}$$

Using $\mathrm{D}_t U_i = -U_l \partial_i v^l$ from (10), the right hand side simplifies greatly by cancellation, and we finally obtain

$$\mathrm{D}_t \boldsymbol{w} = -\, Z^l (\partial_k U_l - \partial_l U_k)\, \nabla v^k = -w_k\, \nabla v^k\, , \tag{29}$$

Substituting (26), (27) and (29) into (25), we obtain

$$\begin{aligned} \mathrm{D}_t\boldsymbol{v} &= \nabla(\mathrm{D}_t\phi) + s\,\nabla(\mathrm{D}_t\psi) - (\partial_k\phi + s\partial_k\psi + w_k)\nabla v^k \\ &= \nabla(\mathrm{D}_t\phi) + s\,\nabla(\mathrm{D}_t\psi) - v_k\,\nabla v^k = \nabla(\mathrm{D}_t\phi + s\,\mathrm{D}_t\psi - \tfrac{1}{2}\,v^2) + T\,\nabla s. \end{aligned} \tag{30}$$

where (15) was used to obtain the last equality. In view of $\mathrm{D}_t = \partial_t + \boldsymbol{v}\cdot\nabla$, the expression in the first ∇-operator of the last expression (30) can be rewritten as

$$\begin{aligned} &\mathrm{D}_t\phi + s\,\mathrm{D}_t\psi - \tfrac{1}{2}\,v^2 = \partial_t\phi + s\,\partial_t\psi + \boldsymbol{v}\cdot\nabla\phi + \boldsymbol{v}\cdot(s\nabla\psi) - \tfrac{1}{2}\,v^2 \\ =\;& \partial_t\phi + s\,\partial_t\psi + \boldsymbol{v}\cdot(\boldsymbol{v}-\boldsymbol{w}) - \tfrac{1}{2}\,v^2 = \partial_t\phi + s\,\partial_t\psi + \tfrac{1}{2}\,v^2 - \boldsymbol{v}\cdot\boldsymbol{w}, \end{aligned} \tag{31}$$

where $\nabla\phi + s\nabla\psi = \boldsymbol{v} - \boldsymbol{w}$ was used in the second equality. Using (13), the equation (30) reduces to Euler's equation of motion (2):

$$\mathrm{D}_t\boldsymbol{v} = -\nabla\,(h + \Psi_e) + T\,\nabla s = -\frac{1}{\rho}\,\nabla p - \nabla\,\Psi_e\,. \tag{32}$$

since $\mathrm{d}h - T\,\mathrm{d}s = (1/\rho)\,\mathrm{d}p$. The last is the standard form of Euler's equation of motion.

Thus, it is found that Euler's equation of motion is satisfied by the set of equations (11) $\sim$ (20) representing *rotational compressible* flow field of an ideal fluid.

In the case of isentropic flows of uniform s, the equation (32) becomes

$$\partial_t\boldsymbol{v} + (\nabla\times\boldsymbol{v})\times\boldsymbol{v} + \nabla(\tfrac{1}{2}\,v^2) = -\nabla\,(h + \Psi_e)\,. \tag{33}$$

Taking curl of (33) and setting $\nabla\times\boldsymbol{v} = \boldsymbol{\omega}$, we obtain the vorticity equation:

$$\partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = 0. \tag{34}$$

4. Variational formulation

The general representations considered so far in §2 and 3 can be derived from the action principle. The Lagrangian functional consists of main terms of total kinetic energy, internal energy ϵ (with negative sign) and force potential Ψ_e (with negative sign), which are supplemented with three terms, *i.e.* two terms yielding the equations of continuity and entropy and the third term yielding a new *rotational* velocity component $\boldsymbol{w}$. Precisely, the total Lagrangian L, Lagrangian density Λ‖, and the action J are defined by

$$L = \int_V \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{U}, \boldsymbol{Z})\;\mathrm{d}^3\boldsymbol{x}\,, \tag{35}$$

$$\begin{aligned} \Lambda &= \tfrac{1}{2}\,\rho\,\langle\boldsymbol{v},\boldsymbol{v}\rangle - \rho\,\epsilon(\rho, s) - \rho\,\Psi_e + \phi\,(\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\,) \\ &\qquad + \psi\,(\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\,) - \rho\,\langle\mathcal{L}_t^*[\boldsymbol{U}],\,\boldsymbol{Z}\rangle, \end{aligned} \tag{36}$$

$$J = \int_{t_1}^{t_2} L\,\mathrm{d}t = \int_{t_1}^{t_2}\mathrm{d}t\int_V \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{U}, \boldsymbol{Z})\;\mathrm{d}^3\boldsymbol{x}, \tag{37}$$

$$\nabla\cdot\boldsymbol{U} = 0, \qquad \nabla\cdot\bar{\boldsymbol{Z}} = 0 \qquad (\bar{\boldsymbol{Z}} = \rho\boldsymbol{Z}). \tag{38}$$

‖ The two terms $\phi\,\big(\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\big)$ and $\psi\,\big(\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\big)$ in the definition (36) can be replaced by $-\rho\,\mathrm{D}_t\phi$ and $-\rho s\,\mathrm{D}_t\psi$ respectively. See Kambe (2008, 2010) for the latter case. Present definition of Λ is used here because that is the case of Clebsch-Salmon considered in §5 (*iii*).

where V is a flow domain in the $\boldsymbol{x}$-space, and both of V and $I_t = [t_1,\, t_2]$ (time interval) are chosen arbitrarily. In regard to the scalar product $\langle \cdot,\, \cdot \rangle$, see the last paragraph of §1 and the footnote there (note that $v_i = v^i$ in the euclidian space). The field variables are the same as defined in the previous sections. The last term of Λ, $\rho \langle \mathcal{L}_t^*[\boldsymbol{U}],\, \boldsymbol{Z} \rangle$, is a newly introduced term in order to incorporate rotational component into the flow field naturally by means of the variational formulation. It will be shown in §6 (a) that each of the last three terms of Λ can be expressed in the form of total time derivative D_t, so that they do not influence the Euler-Lagrange equation derived from the first three main terms (Kambe 2008, 2010).

4.1. Principle of least action

The action principle is defined by vanishing of the variation of J with respect to arbitrary variations of all the variables $\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{U}$, and $\boldsymbol{Z}$, where all the variations are assumed to be independent, and also to vanish on the boundary surface Σ enclosing the integration domain $V \otimes I_t$. Substituting the varied variables $v^i + \delta v^i$, $\rho + \delta\rho$, $s + \delta s$, $\phi + \delta\phi$, $\psi + \delta\psi$, $U_i + \delta U_i$ and $Z^i + \delta Z^i$, into $\Lambda(v^i, \rho, s, \phi, \psi, U_i, Z^i)$, its variation $\delta\Lambda$ can be written as follows:

$$\begin{aligned} \delta\Lambda = \Lambda_{v^i}\, \delta v^i + \Lambda_\rho\, \delta\rho + \Lambda_s\, \delta s + \Lambda_\phi\, \delta\phi + \Lambda_\psi\, \delta\psi \\ + \Lambda_{Z^i} \cdot \delta Z^i + \Lambda_{U_i} \cdot \delta U_i + \partial_t(\Lambda_t) + \partial_i(\Lambda_i), \end{aligned}$$

We substitute this into $\delta J = \int\!\int \delta\Lambda\ \mathrm{d}t\mathrm{d}^3\boldsymbol{x} = 0$. First, we consider the last two divergence terms $\partial_t(\Lambda_t) + \partial_i(\Lambda_i)$ of $\delta\Lambda$, which are transformed to vanishing surface integrals owing to the assumed boundary conditions.

Next, consider the first three terms of the first line and the first term of the second line. From the principle $\delta J = 0$, we must have $\Lambda_{v^i} = 0$, $\Lambda_\rho = 0$, $\Lambda_s = 0$ and $\Lambda_{Z^i} = 0$ for independent variations δv^i, $\delta\rho$, δs, and δZ^i. From the definition (36) of Λ, we have

$$\Lambda_{v^i} = \rho\left(v_i - \partial_i\phi - s\,\partial_i\psi\right) + \rho\Big(-Z^k\,\partial_i U_k + Z^k\,\partial_k U_i\Big) = 0\,, \tag{39}$$

$$\Lambda_\rho = \tfrac{1}{2}v^2 - h - \Psi_e - \mathrm{D}_t\phi - s\,\mathrm{D}_t\psi - \langle \mathcal{L}_t^*[\boldsymbol{U}],\, \boldsymbol{Z}\rangle + const = 0\,,$$

$$\Lambda_s = \rho\,(\,\mathrm{D}_t\psi + T\,) = 0\,,$$

$$\Lambda_{Z^i} = -\rho\,(\mathcal{L}_t^*[\boldsymbol{U}])_i = 0\,. \tag{40}$$

where we used the relation $\delta(\rho\epsilon(\rho, s)) = h\,\delta\rho + \rho T\,\delta s$ obtained from the standard thermodynamics.¶ The second term of Λ_{v^i} is derived from the last term $-\rho\,(\mathcal{L}_t^*[\boldsymbol{U}])_k\, Z^k$ of Λ defined by (36), which is

$$\begin{aligned} -\rho\, Z^k\,(\mathcal{L}_t^*[\boldsymbol{U}])_k &= -\rho\, Z^k\Big(\partial_t U_k + v^i\partial_i U_k + U_i\partial_k v^i\Big) \\ &= -\rho\, Z^k\Big(\partial_t U_k + v^i\partial_i U_k - v^i\partial_k U_i\Big) - \partial_k\Big(\rho\, Z^k\, U_i\, v^i\Big), \end{aligned} \tag{41}$$

where $\partial_k(\rho\, Z^k) = 0$ was used in the last equality. The coefficient of v^i in the first term gives the second of (39), where $-Z^k\,\partial_i U_k + Z^k\,\partial_k U_i = -(\boldsymbol{Z} \times (\nabla \times \boldsymbol{U}))_i$. A constant

¶ $(\partial\epsilon/\partial\rho)_s = p/\rho^2$, $(\partial/\partial\rho)_s(\rho\,\epsilon) = \epsilon + \rho\,(\partial\epsilon/\partial\rho)_s = \epsilon + p/\rho = h$, and $(\partial\epsilon/\partial s)_\rho = T$. Then we have $\mathrm{d}\epsilon = (p/\rho^2)\,\mathrm{d}\rho + T\mathrm{d}s$ and $\mathrm{d}h = (1/\rho)\,\mathrm{d}p + T\mathrm{d}s$.

term is included in the expression of Λ_ρ because the potential Ψ_e can have an arbitrary constant. Thus we obtain

$$\boldsymbol{v} = \nabla\,\phi + s\,\nabla\psi + \boldsymbol{w}, \qquad \boldsymbol{w} \equiv \boldsymbol{Z} \times (\nabla \times \boldsymbol{U})\,, \tag{42}$$

$$\mathrm{D}_t\phi + s\,\mathrm{D}_t\psi - \tfrac{1}{2}\,v^2 + h + \Psi_e = \mathit{const}\ \ , \tag{43}$$

$$\mathrm{D}_t\psi + T = 0\,, \qquad \mathcal{L}_t^*[\boldsymbol{U}] = 0\,. \tag{44}$$

It is noted that the representation of $\boldsymbol{v}$ of (42) is given in the forms of *covariant* vector field. Namely, first two terms are obvious because ∇ applied to a scalar function is a covariant vector. In differential geometry, a scalar product $w_i v^i$ is a pairing of a covariant vector w_i and a contravariant vector v^i, *i.e.* equivalently a pairing of a tangent vector v^i and a cotangent vector w_i in the terminology of present paper. The scalar product is meant to be invariant with respect to coordinate transformations. The third term $\boldsymbol{w} = \boldsymbol{Z} \times (\nabla \times \boldsymbol{U})$ is also a covariant vector $w_i = Z^k(\partial_i U_k - \partial_k U_i)$, because this is obtained from variation with respect to v^i of the second and third terms in the first parentheses of (41): $-\rho\, v^i\, Z^k(\partial_i U_k - \partial_k U_i) = -\rho\, v^i\, w_i$, which is a pairing of a tangent vector v^i and a cotangent vector w_i, multiplied by $-\rho$.

Replacing the first three terms of (43) with (31), we obtain [+]

$$\tfrac{1}{2}\,v^2 + h + \Psi_e + \partial_t\phi + s\,\partial_t\psi - \boldsymbol{v}\cdot\boldsymbol{w} = 0. \tag{45}$$

In regard to the variations of Λ with respect to the variations $\delta\phi$ and $\delta\psi$, we obtain

$$\delta\phi: \quad \Lambda_\phi\,\delta\phi, \qquad \Lambda_\phi \equiv \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}),$$

$$\delta\psi: \quad \Lambda_\psi\,\delta\psi, \qquad \Lambda_\psi \equiv \partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}).$$

Then, we must have $\Lambda_\phi = 0$ and $\Lambda_\psi = 0$ by the variational principle. The first results in the continuity equation:

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\,. \tag{46}$$

Since $\Lambda_\psi = s\,\Lambda_\phi + \rho\mathrm{D}_t s = \rho\mathrm{D}_t s$, the second results in the entropy equation:

$$\mathrm{D}_t s \equiv \partial_t s + \boldsymbol{v}\cdot\nabla s = 0\,. \tag{47}$$

The remaining last term is the Λ-variation with respect to the variation δU_i:

$$\delta U_i: \quad \Lambda_{U_i}\,\delta U_i - \partial_t(\bar{Z}^i\,\delta U_i) - \partial_k(v^k U_i \bar{Z}^i), \qquad \Lambda_{U_i} \equiv \mathcal{L}_t[\bar{Z}^i] + \bar{Z}^i\partial_k v^k.$$

where $\bar{Z}^i = \rho Z^i$. We must have $\Lambda_{U_i} = 0$. Thus,

$$\begin{aligned} 0 &= \mathcal{L}_t[\bar{Z}] + \bar{Z}\partial_k v^k = \mathcal{L}_t[\rho\boldsymbol{Z}] + \boldsymbol{Z}\,\rho(\nabla\cdot\boldsymbol{v}) \\ &= \rho\,\mathcal{L}_t[\boldsymbol{Z}] + \boldsymbol{Z}\,(\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})) = \rho\,\mathcal{L}_t[\boldsymbol{Z}], \end{aligned} \tag{48}$$

where the definition of $\mathcal{L}_t$ of (10) was used to obtain the second line, and (46) was used in the last equality. Thus, we obtain

$$\mathcal{L}_t[\boldsymbol{Z}] = \partial_t\boldsymbol{Z} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{Z} - (\boldsymbol{Z}\cdot\nabla)\boldsymbol{v} = 0\,, \tag{49}$$

whereas the vector potential $\bar{\boldsymbol{Z}}$ with the bar satisfies the equation (23).

[+] The right hand side may be a constant because the force is defined by space derivatives of Ψ_e.

Thus, we have obtained the results (42), (44), and (45) $\sim$ (48), from the variational principle under the Lagrangian density Λ of (36) and the divergence-free conditions, $\nabla \cdot \boldsymbol{U} = 0$ and $\nabla \cdot \bar{\boldsymbol{Z}} = 0$, of (38). We have recovered all the equations (11) $\sim$ (19) of general representation of a family of solutions to the Euler equation (5).

4.2. Rotational component

The present *improved* solution includes the new term $\rho\,\langle \mathcal{L}_t^*[\boldsymbol{U}],\, \boldsymbol{Z}\rangle$ in the Lagrangian density Λ, which yields the *rotational* component $\boldsymbol{w}$ of velocity. In a fluid of constant entropy s_0, the velocity $\boldsymbol{v}$ of (42) reduces to

$$\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}, \qquad \Phi = \phi + s_0\psi, \qquad \boldsymbol{w} = \boldsymbol{Z}\times(\nabla\times\boldsymbol{U}). \tag{50}$$

The vorticity curl$\boldsymbol{v}$ is given by $\boldsymbol{\omega}_w \equiv \nabla\times\boldsymbol{w}$:

$$\boldsymbol{\omega}_w = (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{Z} - (\boldsymbol{Z}\cdot\nabla)\boldsymbol{\Omega} - (\nabla\cdot\boldsymbol{Z})\boldsymbol{\Omega}, \qquad \boldsymbol{\Omega} = (\nabla\times\boldsymbol{U}). \tag{51}$$

In §3, we obtained the equation (29), $\mathrm{D}_t\boldsymbol{w} = -w_k\,\nabla v^k$, which can be rewritten as

$$\partial_t\boldsymbol{w} + (\nabla\times\boldsymbol{w})\times\boldsymbol{v} = \nabla(\boldsymbol{w}\cdot\boldsymbol{v}).$$

Taking curl of this equation, we obtain

$$\partial_t\boldsymbol{\omega}_w + \nabla\times(\boldsymbol{\omega}_w\times\boldsymbol{v}) = 0. \tag{52}$$

Thus, it is found that the component $\boldsymbol{w}$ is in fact the *rotational* component which satisfies the vorticity equation (34). This is one of the targets at which the present formulation aimed.

In order to investigate the mathematical form of the rotational component $\boldsymbol{w} = \boldsymbol{Z}\times(\nabla\times\boldsymbol{U})$, we make local analysis of a general velocity field $\boldsymbol{v}(\boldsymbol{x}_0+\boldsymbol{s})$ in the neighborhood of a space point $\boldsymbol{x}_0$. Let us define the relative velocity by $\delta\boldsymbol{v}(\boldsymbol{s}) = \boldsymbol{v}(\boldsymbol{x}_0+\boldsymbol{s}) - \boldsymbol{v}(\boldsymbol{x}_0)$ for a displacement $\boldsymbol{s}$. According to the elementary course of fluid mechanics (*e.g.* Batchelor 1967, or Kambe 2007a), the relative velocity $\delta\boldsymbol{v}(\boldsymbol{s}) = \delta v_i(s^j)$ is expanded with respect to a small displacement $\boldsymbol{s} = (s^j)$. Keeping linear terms only, we obtain

$$\delta v_i^{(a)} = a_{ij}\,s^j, \qquad a_{ij} = -\tfrac{1}{2}\,(\partial_i v_j - \partial_j v_i)\Big|_{x_0}.$$

for local rotational component of anti-symmetric tensor a_{ij}, while the pure straining component $e_{ij}\,s^j$ of a symmetric tensor $e_{ij} = \frac{1}{2}\,(\partial_i v_j + \partial_j v_i)|_{x_0}$ is neglected. Writing as $a_{ij} = -\frac{1}{2}\,\epsilon_{ijk}\,\omega_k^*$ (with ϵ_{ijk} the usual third-order skew-symmetric tensor), we have

$$\delta v_i^{(a)} = -\tfrac{1}{2}\,\epsilon_{ijk}\,\omega_k^*\,s^j, \qquad \delta\boldsymbol{v}^{(a)} = \left(\delta v_i^{(a)}\right) = \tfrac{1}{2}\,\boldsymbol{\omega}_0\times\boldsymbol{s}. \tag{53}$$

where $\boldsymbol{\omega}_0 = (\omega_i^*) = \nabla\times\boldsymbol{v}|_{x_0}$ is the vorticity at $\boldsymbol{x}_0$. This component $\delta\boldsymbol{v}^{(a)}$ represents a rotational motion of angular velocity $\frac{1}{2}\,\boldsymbol{\omega}_0$ around an origin $(\boldsymbol{x}_0)$.

It is interesting to observe that the present expression $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{Z}\times(\nabla\times\boldsymbol{U}) = -\boldsymbol{\Omega}\times\boldsymbol{Z}$ gives a form analogous to $\boldsymbol{\omega}\times\boldsymbol{s}$. In fact, in the neighborhood of a point $\boldsymbol{x}_0$, the vector $\boldsymbol{Z}$ at a point $\boldsymbol{x} = \boldsymbol{x}_0+\boldsymbol{s}$ may be written as $\boldsymbol{Z}_0+\delta\boldsymbol{Z}$ where $\boldsymbol{Z}_0 = \boldsymbol{Z}(\boldsymbol{x}_0)$ and $\delta\boldsymbol{Z} = \partial_j\boldsymbol{Z}\,s^j$. We may compare the pair $(-\boldsymbol{\Omega}, \delta\boldsymbol{Z})$ to $(\frac{1}{2}\,\boldsymbol{\omega}_0, \boldsymbol{s})$, and assume that the field $\boldsymbol{\Omega}$ is nearly constant in the space domain under consideration. Then, the velocity $\boldsymbol{w}$ at a point $\boldsymbol{x}$ is expanded as follows:

$$\boldsymbol{w}(\boldsymbol{x}_0+\boldsymbol{s}) = \boldsymbol{w}_0 - \boldsymbol{\Omega}\times\delta\boldsymbol{Z}, \qquad \boldsymbol{w}_0 = -\boldsymbol{\Omega}\times\boldsymbol{Z}_0.$$

The local motion $-\boldsymbol{\Omega}\times\delta\boldsymbol{Z}$ is like a rotation-like flow of angular velocity $-\boldsymbol{\Omega} = -\nabla\times\boldsymbol{U}$ around an origin located at $\boldsymbol{s}-\delta\boldsymbol{Z}$.

5. New aspects of the present representation

Present formulation is new at least in the following two aspects.
(i) A new point becomes clear if we neglect the third term $\boldsymbol{w}$ of velocity (42) derived from the last term $-\langle\mathcal{L}_t^*[\boldsymbol{U}],\,\rho\boldsymbol{Z}\rangle$ of the Lagrangian density (36). In this case, the velocity is given by $\boldsymbol{v}_0 = \nabla\phi + s\,\nabla\psi$. Taking its curl, the vorticity is given by $\boldsymbol{\omega}_0 = \nabla\times\boldsymbol{v}_0 = \nabla s\times\nabla\psi$. Then the helicity is given as follows:

$$\begin{aligned} H &= \int_V \boldsymbol{v}_0\cdot\boldsymbol{\omega}_0\,\mathrm{d}^3\boldsymbol{x} = \int(\nabla\phi + s\,\nabla\psi)\cdot(\nabla s\times\nabla\psi)\mathrm{d}^3\boldsymbol{x} \\ &= \int_V(\nabla\phi)\cdot\boldsymbol{\omega}_0\,\mathrm{d}^3\boldsymbol{x} = \int\nabla\cdot\big(\phi\,\boldsymbol{\omega}_0\big)\mathrm{d}^3\boldsymbol{x} = \int_S\phi\,\boldsymbol{\omega}_0\cdot\boldsymbol{n}\,\mathrm{d}S = 0, \end{aligned}$$

(where $\nabla\cdot\boldsymbol{\omega}_0 = 0$ is used in the second line) if $\boldsymbol{\omega}_0 = 0$ on the surface S bounding the volume V, or if $|\phi\boldsymbol{\omega}_0| = O(|\boldsymbol{x}|^{-3-\alpha})$ (with a positive parameter α) as $|\boldsymbol{x}|\to\infty$ when V is unbounded space.

Furthermore, if the fluid is isentropic, *i.e.* $s = s_0$ (constant), we have $\boldsymbol{\omega}_0 = 0$. Namely the flow is irrotational without the term $\boldsymbol{w}$. However, in the present solution, we have $\mathrm{curl}\,\boldsymbol{w}\neq 0$ in general, and the flow is *rotational* even in the isentropic fluid. In addition, the helicity does not vanish in the present case.

(ii) Let us consider the significance of the term $\langle\mathcal{L}_t^*[\boldsymbol{U}],\,\boldsymbol{Z}\rangle$ of Λ which yields the rotational component of $\boldsymbol{v}$. It is remarkable that the term satisfies both of translation and rotation symmetries. First, it takes a form of scalar product, which is invariant with respect to rotation of local frame of reference, *i.e.* it has the gauge symmetry with respect to local rotation (Kambe 2010 §7.8 and 7.13). In addition, we can show that $\langle\mathcal{L}_t^*[\boldsymbol{U}],\,\boldsymbol{Z}\rangle$ can be written as $\mathrm{D}_t\langle\boldsymbol{U},\,\boldsymbol{Z}\rangle$. In fact, rewriting $\langle\boldsymbol{U},\,\boldsymbol{Z}\rangle$ as $U_i\,Z^i$, we obtain

$$\mathrm{D}_t\langle\boldsymbol{U},\,\boldsymbol{Z}\rangle = \mathrm{D}_t(U_i\,Z^i) = (\mathcal{L}_t^*[\boldsymbol{U}])_i\,Z^i + U_i\,(\mathcal{L}_t[\boldsymbol{Z}])^i = \langle\mathcal{L}_t^*[\boldsymbol{U}],\,\boldsymbol{Z}\rangle.$$

The second equality can be verified directly by substituting the definitions of $\mathcal{L}_t$ and $\mathcal{L}_t^*$ of (10). (See Frankel (1997, Ch.4) for its equality in general mathematical formulation.) The third equality holds due to (49), which is obtained from the fact that $\boldsymbol{U}$ is an independent field variation. The last expression vanishes by (40) obtained from the variational principle. The equality $\langle\mathcal{L}_t^*[\boldsymbol{U}],\,\boldsymbol{Z}\rangle = \mathrm{D}_t\langle\boldsymbol{U},\,\boldsymbol{Z}\rangle$ means that, not only $\langle\mathcal{L}_t^*[\boldsymbol{U}],\,\boldsymbol{Z}\rangle$ has the rotation symmetry, but also it has the translation symmetry as well, since the operator D_t has local gauge invariance with respect to translation (Kambe (2010) §7.6).

In the traditional formulation, there is a freedom (which is not recognized explicitly) in the transformation between the Lagrangian particle coordinates (a^1, a^2, a^3) and

Eulerian space coordinates (x^1, x^2, x^3). Namely, the relation between them is determined only up to an unknown rotation. This point is detailed in the previous paper (Kambe 2008, §4.2 and 8.2). In fact, the transformation between both spaces is determined locally by nine elements of the matrix $\partial x^k/\partial a^l$. For its purpose, transformation relations of the three vectors, *i.e.* velocity, acceleration and vorticity, suffice to determine the nine elements. This implies the importance of vorticity as an independent field variable.

There is a slight difference in the present term $\langle \mathcal{L}_t^*[\boldsymbol{U}],\, \bar{\boldsymbol{Z}}\rangle$ (where $\bar{\boldsymbol{Z}} = \rho\boldsymbol{Z}$) from that of previous one (Kambe 2008), which is given by the form $\langle \mathcal{L}_t^*[\boldsymbol{U}],\, \boldsymbol{\omega}\,\rangle$. The second factor $\boldsymbol{\omega}$ of the latter is the vorticity $\mathrm{curl}\,\boldsymbol{v}$ itself satisfying the equation (34), whereas the factor $\bar{\boldsymbol{Z}}$ in the former is a vector field governed by (23) which is the same as (34). The field $\bar{\boldsymbol{Z}}(t,\boldsymbol{x})$ is determined by initial condition chosen appropriately. Owing to the new form $\langle \mathcal{L}_t^*[\boldsymbol{U}],\, \bar{\boldsymbol{Z}}\rangle$ of the present study which is linear with respect to $\boldsymbol{v}$, the velocity $\boldsymbol{v}$ is given an explicit form of (42), whereas it is expressed implicitly in the previous formulation.

(*iii*) According to (40) and (48) and, the potentials U_i (a cotangent vector) and Z^i (a tangent vector) satisfy the following equations:

$$\mathrm{D}_t U_i = -U_k \partial_i v^k, \qquad \mathrm{D}_t Z^i = Z^k \partial_k v^i. \tag{54}$$

These equations are essentially different in character from the equations of the Clebsch potentials. In fact, the Clebsch-Salmon solution (Appendix A) is expressed as

$$\text{Lagrangian density} : \Lambda_{\mathrm{C}} = \rho\Big(\tfrac{1}{2}\,|\boldsymbol{v}|^2 - e(\rho,s) - \Psi_e - \sum_i \eta_i\,\mathrm{D}_t\zeta_i\Big)$$

$$+\,\phi\,(\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\,) + \psi\,(\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\,), \tag{55}$$

$$\text{Velocity} \quad : \quad \boldsymbol{v} = \nabla\phi + s\,\nabla\psi + \sum_i \eta_i\,\nabla\zeta_i, \tag{56}$$

$$\mathrm{D}_t\eta_i = 0, \qquad \mathrm{D}_t\zeta_i = 0, \qquad (i = 1,2). \tag{57}$$

and $\mathrm{D}_t s = 0$, $\mathrm{D}_t\psi = 0$ (Clebsch 1859; Salmon 1988; Kambe 2010). This Clebsch-Salmon solution satisfies Euler's equation of motion (Appendix A). Comparing the two equations (54) and (57), it is obvious that the right hand sides are different, and that the right hand sides of (57) vanish. Firstly, the equation (54) presupposes existence of a tangent vector $\boldsymbol{Z} = (Z^i)$ and a cotangent vector $\boldsymbol{U} = (U_i)$, and in addition assures the invariance of the scalar product $U_i Z^i$ (see (*ii*) and the footnote of §1). On the other hand, the Lagrangian density (55) of the Clebsch-Salmon formulation relies only on cotangent vectors such as $\nabla\zeta_i$, and the potentials η_i and ζ_i are simply convected by the flow as scalars. A certain vectorial property including scalar product of their pairing (rotational symmetry) is not imposed.

6. Physical and mathematical significance

The present formulation has some characteristic features, which are discussed in this section from physical and mathematical point of view.

(a) The Lagrangian density Λ is defined by (36), which is reproduced here:

$$\begin{aligned}\Lambda = \tfrac{1}{2}\,\rho\,\langle \boldsymbol{v},\boldsymbol{v}\rangle - \rho\,\epsilon(\rho,s) - \rho\,\Psi_e + \phi\,(\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\,) \\ +\,\psi\,(\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\,) - \rho\,\langle \mathcal{L}_t^*[\boldsymbol{U}],\,\boldsymbol{Z}\rangle.\end{aligned}$$

Physical significance of the present form is as follows. The first three terms $\rho(\frac{1}{2}\,|\boldsymbol{v}|^2 - \epsilon - \Psi_e)$ of Λ constitute the main structure of the Lagrangian. Let us denoteit as Λ_m. The Euler-Lagrange equation derived from the variation of Λ_m reduces to Euler's equation of motion (Herivel 1955; Eckart 1960; Seliger & Whitham 1968).

Next two terms can be expressed as

$$\begin{aligned}\phi\,(\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v})\,) \quad &= \; -\,\rho\,\mathrm{D}_t\phi + \partial_t(\rho\phi) + \nabla\cdot(\rho\phi\,\boldsymbol{v}),\\ \psi\,(\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v})\,) &= \; -\,\rho\,\mathrm{D}_t(s\psi) + \partial_t(\rho s\psi) + \nabla\cdot(\rho s\psi\,\boldsymbol{v}),\\ \Big(\rho\mathrm{D}_t\phi + \rho s\mathrm{D}_t\psi\Big)\,\mathrm{d}V &= \mathrm{D}_t\Big[\phi + s\,\psi\Big]\,\rho\,\mathrm{d}V,\end{aligned}$$

since $\mathrm{D}_t s = 0$. The last term is rewritten as

$$-\rho\,\langle \mathcal{L}_t^*[\boldsymbol{U}],\boldsymbol{Z}\rangle = -\rho\,\mathcal{L}_t\Big[\langle \boldsymbol{U},\,\boldsymbol{Z}\rangle\Big] = -\rho\,\mathrm{D}_t\Big[\langle \boldsymbol{U},\,\boldsymbol{Z}\rangle\Big],$$

where the first equality is due to (19) and the footnote of §1, and the second equality holds by the identity $\mathcal{L}_t[F] = \mathrm{D}_t[F]$ for a scalar field $F(t,\boldsymbol{x})$. Thus, the last three terms of Λ can be written in the form,

$$-\rho\,\mathrm{D}_t\Big[\phi + s\,\psi + \langle \boldsymbol{U},\,\boldsymbol{Z}\rangle\Big] + \partial_t\,(\cdots) + \nabla\cdot(\cdots), \tag{58}$$

which can be integrated in the action integral (37) of the form $\int\int\,\mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}$. For example, consider the following integration $\int\mathrm{d}t\int\rho\,\mathrm{D}_t F\,\mathrm{d}^3\boldsymbol{x}$, which is transformed as follows:

$$\begin{aligned}\int_{I_t}\mathrm{d}t\int_V\Big(\rho\,\partial_t F + \rho\,v^k\partial_k F\Big)\,\mathrm{d}^3\boldsymbol{x} &= \int\mathrm{d}t\int\Big(\partial_t(\rho F) + \partial_k(\rho v^k F) - F\,[\partial_t\rho + \partial_k(\rho v^k)\,]\Big)\mathrm{d}^3\boldsymbol{x}\\ &= \int_{I_t}\mathrm{d}t\int_V\Big(\partial_t(\rho F) + \partial_k(\rho v^k F)\Big)\,\mathrm{d}^3\boldsymbol{x}\end{aligned}$$

Therefore, the three terms of (58) are transformed to integrals over the boundary surface Σ enclosing the domain $V\otimes I_t$, and they do not influence the Euler-Lagrange equation derived from the variation of Λ_m. However, those terms give non-trivial contribution in the expressions of Eulerian representation, such as the expressions of (11) $\sim$ (20).

(b) From the set of equations (11) $\sim$ (20), time evolutions of ρ, s, ϕ, ψ, $\boldsymbol{U}$ and $\boldsymbol{Z}$ are determined. The variables to determine $\boldsymbol{v}$ are ϕ, s, ψ, $\boldsymbol{U}$ and $\boldsymbol{Z}$, governed by (13), (15), and (17)$\sim$(19). Once ϕ, s, ψ, $\boldsymbol{U}$ and $\boldsymbol{Z}$ are known, the velocity $\boldsymbol{v}$ is determined by (11). The density ρ is determined by (14). The pressure p is determined by $p = -(\partial\epsilon/\partial\rho^{-1})_s$ once the thermodynamic equation of state $\epsilon = \epsilon(\rho,s)$ is given. Then, the temperature and enthalpy are given by $T = (\partial\epsilon/\partial s)_\rho$ and $h = \epsilon + p/\rho$.

In the case of isentropic flows, s is a constant s_0 and the velocity can be written as $\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}$ where $\Phi = \phi + s_0\psi$ (see (50)). Hence, the variables s and ψ drop from the system. Correspondingly, the two equations (15) and (16) drop.

Furthermore, if the fluid density is a constant ρ_0 in addition to $s = s_0$, the continuity equation (14) reduces to the solenoidal condition $\nabla\cdot\boldsymbol{v} = 0$. This requires that the velocity potential Φ should satisfy (i) $\nabla^2\Phi = -\nabla\cdot\boldsymbol{w}$, or (ii) $\nabla^2\Phi = 0$ and $\nabla\cdot\boldsymbol{w} = 0$.

(c) In the case of (ii) of uniform density ρ_0 (and uniform entropy), the velocity is represented by $\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}$, where $\nabla^2\Phi = 0$, $\boldsymbol{w} = \boldsymbol{Z}\times\boldsymbol{\Omega}$ and $\boldsymbol{\Omega} \equiv \nabla\times\boldsymbol{U}$, and

$$\mathcal{L}_t^*[\boldsymbol{U}] = 0, \qquad \mathcal{L}_t[\boldsymbol{Z}] = 0, \tag{59}$$

under
$$\mathrm{div}\,\Big(\boldsymbol{Z}\times\boldsymbol{\Omega}\Big) = 0, \quad \partial_i U_i = 0, \quad \partial_k(\rho Z)^k = 0. \tag{60}$$

The vector $\boldsymbol{w}$ is determined by $\boldsymbol{\Omega} = \nabla\times\boldsymbol{U}$ and $\boldsymbol{Z}$. Total six components of $\boldsymbol{U}$ and $\boldsymbol{Z}$ are constrained by the three conditions of (60), so that independent components are three. Therefore, once three components from $\boldsymbol{Z}$ and $\boldsymbol{\Omega}$ are given initially at $t = 0$ and in addition three initial values of $\boldsymbol{w}_0$ are given, then initial values of $\boldsymbol{\Omega}$ and $\boldsymbol{Z}$ can be determined in principle by the equation $\boldsymbol{Z}\times\boldsymbol{\Omega} = \boldsymbol{w}$ (where $\boldsymbol{w}\perp\boldsymbol{Z}$, $\boldsymbol{\Omega}$).

(d) For example, suppose that we have $\rho = \rho_0$, $s = s_0$ and $\nabla^2\Phi = 0$ with no external force $\Psi_e = 0$, and in addition, suppose that initial value $\boldsymbol{Z}_0$ is given for $\boldsymbol{Z}$, then initial value of $\boldsymbol{\Omega}$ is determined, by using the property $\boldsymbol{Z}_0\cdot\boldsymbol{w}_0 = 0$, as

$$\boldsymbol{\Omega}_0 = \boldsymbol{w}_0\times\boldsymbol{Z}_0/|\boldsymbol{Z}_0|^2 + \boldsymbol{\Omega}_0^{\parallel}. \tag{61}$$

There is arbitrariness in $\boldsymbol{\Omega}_0$ for the component $\boldsymbol{\Omega}_0^{\parallel}$ parallel to $\boldsymbol{Z}_0$.

Subsequent development can be found by the above equations of (59) and (60). Once the velocity is determined, the pressure $p = \rho_0\, h + const$ (since $\mathrm{d}h = \mathrm{d}p/\rho_0$ for $\rho = \rho_0$ and $s = s_0$) is determined by the following equation obtained from (13):

$$p/\rho_0 + \tfrac{1}{2}\,v^2 + \partial_t\Phi - \boldsymbol{v}\cdot\boldsymbol{w} = const\ . \tag{62}$$

7. Example

We consider an example of application of the present formulation. The fluid is assumed to be isentropic with a constant entropy s_0. Its velocity field is expressed by

$$\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}, \qquad \boldsymbol{w} = \boldsymbol{Z}\times\boldsymbol{\Omega}, \qquad \boldsymbol{\Omega} = \nabla\times\boldsymbol{U}, \tag{63}$$

and $\Phi = \phi + s_0\psi$. The fields $\boldsymbol{U}$ and $\boldsymbol{Z}$ are governed by (59) and (60).

(a) *Rankine's vortex*

Rankine's vortex of strength γ is a columnar vortex parallel to the straight (say) z-axis with its cross-section being circular and its radius being a. The z axis of the cylindrical frame (z, r, θ) is taken along the central line of the vortex, and the velocity is represented by $\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}$. The vorticity is given by

$$\boldsymbol{\omega} = (\omega, 0, 0), \qquad \text{where} \quad \omega = \omega_0 \quad (r < a); \qquad 0 \quad (r > a). \tag{64}$$

where ω_0 is a constant, defined by $\gamma/(\pi a^2)$.

The vortex has a cylindrical vortex core, while the flow is irrotational out of the core. Then the velocity field outside the core $(r > a)$ is represented by

$$\boldsymbol{v}_o = \nabla\Phi = \Big(0, 0, \frac{\beta}{r}\Big), \qquad \text{with} \quad \Phi = \beta\,\theta, \quad \beta = \frac{\gamma}{2\pi} \quad \text{and} \quad \boldsymbol{w} = 0. \tag{65}$$

This is equivalent to the velocity of a line vortex (coinciding with z axis).

Within the core $r < a$, we define

$$\boldsymbol{Z} = (\tfrac{1}{2}\,\omega_0,\, 0,\, 0), \qquad \boldsymbol{U} = (0,\, 0,\, U_\theta), \quad U_\theta = -zr. \tag{66}$$

This gives

$$\boldsymbol{\Omega} = (\Omega_z, \Omega_r, \Omega_\theta) = \nabla \times \boldsymbol{U} = (-z,\, r,\, 0).$$

Then the velocity field within the core ($r < a$) is

$$\boldsymbol{v}_i = \boldsymbol{w} = \boldsymbol{Z} \times \boldsymbol{\Omega} = (0,\, -\tfrac{1}{2}\,\omega_0\Omega_\theta,\, \tfrac{1}{2}\,\omega_0\Omega_r) = (0,\, 0,\, \tfrac{1}{2}\,\omega_0 r), \tag{67}$$

where $\Phi = const$. This represents the velocity as if the vortex core is rotating like a solid-body around its axis. Obviously, the azimuthal component v_θ (only non-zero component) of the velocity takes the same value on both sides of the cylinder $r = a$. Namely, $(\boldsymbol{v}_i)_\theta|_{r=a} = \frac{1}{2}\,\omega_0 a = \gamma/(2\pi a)$ is equal to $(\boldsymbol{v}_o)_\theta|_{r=a} = \gamma/(2\pi a)$.

(*b*) *Interaction between a columnar vortex and sound waves*

A system of a vortex interacting with waves in fluid mechanics is analogous to that of the Aharonov-Bohm effect of quantum-mechanical system. Wave-front dislocation in the quantum-mechanical system was studied by Berry *et al.* (1980) with analogy of water waves on horizontal surface interacting with a vertical swirling vortex. This analogy certainly makes sense. However, more appropriate analogy would be the interaction of sound waves with a columnar vortex such as the Rankine's vortex. This implies that a fluid Aharonov-Bohm effect can be studied under the present formulation.

Fluid Aharonov-Bohm effect for water waves on horizontal surface interacting with a vertical swirling vortex was studied mathematically in detail by Coste *et al.* (1999). In regard to sound waves interacting with a columnar vortex, its detailed study remains to be done in future.

8. Summary and discussions

A new representation of rotational flow fields is proposed for an ideal compressible fluid. This is regarded as generalization of the Bernoulli's equation to compressible rotational flows. Usually, the Bernoulli equation is an expression valid along each streamline in steady flows, or one valid at all points in unsteady irrotational flow. Present expression of generalized Bernoulli equation is given by an equation among a set of expressions describing unsteady rotational flows. Remarkably, it is verified that the set of new representations satisfies Euler's equation of motion.

In the last section, Rankine's vortex is given a new expression to show application of the present representation. Another application can be conceivable for the Gross-Pitaevskii equation governing superfluid flows. The equation can be divided into imaginary part and real part. One of them reduces to the continuity equation of the form (14), while the other has a form analogous to (13) (with $s = 0$) of generalized Bernoulli equation. This analogy implies that superfluid flow may be described by the fluid-dynamic equations of an ideal fluid, which will be reported elsewhere.

The new representation conidered in this paper is obtained from the variational principle with the Lagrangian including a new term of scalar product of the form $\langle \mathcal{L}_t^*[\boldsymbol{U}],\, \boldsymbol{Z} \rangle$. This term has the gauge symmetry with respect to both translation and rotation. In fact, this gives an explicit representation of a new rotational velocity component satisfying the vorticity equation. In this sense, this is regarded as generalization of the representation of Clebsch-Salmon. This expression has been found during an effort to establish equivalence between the two versions of Eulerian and Lagrangian forms of the action principle.

Acknowledgments

The author is grateful to the referees for thoughtful review comments which lead to amendment and improvement of the text with a paragraph of Crocco's theorem and a note on covariance of the representation of velocity $\boldsymbol{v}$, and with addition of the last section of examples.

Appendix A. Clebsch-Salmon representation

An integration of the hydrodynamic equation was given by Clebsch (1859) for an ideal incompressible fluid, provided that the external force has a potential Ψ_e. In this solution, the velocity $\boldsymbol{u}$ is represented in general in terms of $2n+1$ scalar functions of $2n+1$ space variables, where the functions depend on Lagrangian particle coordinates only. This is cited by Lamb (1932) as Clebsch's transformation of the hydrodynamic equations for three-dimensional case of $n = 1$. As mentioned in the Introduction, it is known that there is discrepancy between the two approaches of Lagrangian and Eulerian. Lin (1963) and Salmon (1988) tried an effort to resolve it and the latter proposed the following set of representations for a compressible ideal fluid. The Clebsch-Salmon representation (with some modification) is as follows:

$$\boldsymbol{u} = \nabla\phi + s\,\nabla\psi + \sum_{i=1}^{2} \eta_i\,\nabla\zeta_i, \tag{A.1}$$

$$\tfrac{1}{2}\,u^2 + h + \Psi_e + \partial_t\phi + s\partial_t\psi + \sum_{i=1}^{2} \eta_i\,\partial_t\zeta_i = 0, \tag{A.2}$$

$$\mathrm{D}_t s = 0, \qquad \mathrm{D}_t\psi = -(\partial\epsilon/\partial s)_\rho, \qquad \mathrm{D}_t\eta_i = 0, \quad \mathrm{D}_t\zeta_i = 0, \tag{A.3}$$

$$\partial_t\rho + \nabla\cdot(\rho\,\boldsymbol{u}) = 0. \tag{A.4}$$

where s, h and ϵ are the specific entropy, enthalpy and internal energy ($(\partial\epsilon/\partial s)_\rho = T$, the temperature). From the velocity (A.1), we obtain the vorticity $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} = \nabla s\times\nabla\psi + \sum_{i=1}^{2} \nabla\eta_i\times\nabla\zeta_i, \tag{A.5}$$

In addition, we can represent the terms $\boldsymbol{\omega}\times\boldsymbol{u}$ and $\partial_t\boldsymbol{u}$ by

$$\boldsymbol{\omega}\times\boldsymbol{u} = (\boldsymbol{u}\cdot\nabla s)\nabla\psi - (\boldsymbol{u}\cdot\nabla\psi)\nabla s + \sum_{i=1}^{2}\Big\{(\boldsymbol{u}\cdot\nabla\eta_i)\nabla\zeta_i - (\boldsymbol{u}\cdot\nabla\zeta_i)\nabla\eta_i\Big\},$$

$$\partial_t\boldsymbol{u} \quad = \nabla\partial_t\phi + \partial_t s\,\nabla\psi + s\,\nabla\partial_t\psi + \sum_{i=1}^{2}\Big(\partial_t\eta_i\,\nabla\zeta_i + \eta_i\,\nabla\partial_t\zeta_i\Big).$$

Adding these two equations, and using (A.3), we obtain

$$\partial_t\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u} = \nabla\Big(\partial_t\phi + s\,\partial_t\psi + \sum_{i=1}^{2}\eta_i\,\partial_t\zeta_i\Big) - (\mathrm{D}_t\psi)\,\nabla s$$
$$+ (\mathrm{D}_t s)\,\nabla\psi + \sum_{i=1}^{2}\Big\{(\mathrm{D}_t\eta_i)\,\nabla\zeta_i - (\mathrm{D}_t\zeta_i)\,\nabla\eta_i\Big\}.$$

The first term on the right can be replaced by $-\nabla(\frac{1}{2}\,u^2 + h + \Psi_e)$ from (A.2), and the second term by $T\nabla s$, while all the three terms of the second line vanish due to (A.3). Thus, we have

$$\partial_t\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u} = -\nabla\Big(\tfrac{1}{2}\,u^2 + h + \Psi_e\Big) + T\nabla s. \tag{A.6}$$

This is Euler's equation of motion in the form of (5). Because of the equations (A.4) and (A.5), the flow is compressible and rotational in general. Namely, the set of equations (A.1)∼(A.4) solves the Euler equation. It is interesting to see that the expression (A.2) is a generalization of the Bernoulli equation (8).

When $\eta_i = 0$ $(i = 1, 2)$, we have

$$\boldsymbol{u}_C = \nabla\phi + s\,\nabla\psi, \qquad \boldsymbol{\omega}_C = \nabla s\times\nabla\psi. \tag{A.7}$$

This is the representation of Clebsch (1859) as an integral of incompressible flows. This form has a limited generality by the following two reasons. First, the helicity H vanishes. Namely, we have

$$H = \int \boldsymbol{\omega}_C\cdot\boldsymbol{u}_C\,\mathrm{d}V = \int \boldsymbol{\omega}_C\cdot\nabla\phi\,\mathrm{d}V = \int \nabla(\boldsymbol{\omega}_C\,\phi)\,\mathrm{d}V, \tag{A.8}$$

since $s\nabla\psi\cdot(\nabla s\times\nabla\psi) = 0$ and $\nabla\cdot\boldsymbol{\omega}_C = 0$. We obtain $H = 0$, either if $\boldsymbol{\omega} = 0$ at points out of a domain, or if $|\boldsymbol{\omega}|\to 0$ sufficiently rapidly as $|\boldsymbol{r}|\to\infty$. Secondly, if $s = const$ (*i.e.* isentropic), the flow becomes irrotational, *i.e.* $\boldsymbol{\omega}_C = 0$.

The field representation (A.1)∼(A.4) can be derived from the variational principle (*i.e.* Hamilton's principle) by the Lagrangian functional Λ_{C} given by (55) in §5 (*iii*).

The representation $\boldsymbol{u} = \nabla\phi + s\,\nabla\psi + \eta_1\,\nabla\zeta_1$ has a shortage analogous to the Clebsch form (A.7). If $s = const$, the vorticity is $\boldsymbol{\omega} = \nabla\eta_1\times\nabla\zeta_1$, resulting in vanishing helicity by the same calculus as that of (A.8). A traditional resolution to avoid the vanishing helicity is to add another term $\eta_2\,\mathrm{D}_t\zeta_2$, giving $\boldsymbol{u}$ an additional term $\eta_2\,\nabla\zeta_2$. This is an *ad hoc* approach which lacks any physical principle to justify the terms $\sum_i \eta_i\,\mathrm{D}_t\zeta_i$ for introduction of new terms in Λ_{C}. This is a weak point of this formulation.

References

Batchelor G K 1967 *An Introduction to Fluid Dynamics* (Cambridge: Cambridge University Press) p 615

Bernoulli D 1738 *Hydrodynamica* (Strasbourg: Johannes Reinhold Dulsecker); [*also* English translation] Hydrodynamics (by Daniel Bernoulli) and Hydraulics (by Johann Bernoulli) (New York: Dover Publications 1968) p 456

Berry M V, Chambers R G, Large M D , Upstill C and Walmsley 1980 Wavefront dislocations in the Aharonov-Bohm effect and its water wave analogue *Eur. J. Phys.* **1** 154-162.

Bretherton F P 1970 A note on Hamilton's principle for perfect fluids *J. Fluid Mech.* **44** 19-31.

Clebsch A 1859 Über die Integration der hydrodynamischen Gleichungen *J. Reine und Angew. Math.* **56** 1-10

Coste C, Lund F, and Umeki M 1999 Scattering of dislocated wave fronts by vertical vorticity and the Aharonov-Bohm effect, I Shallow water *Phys Rev E* **60** 4908-4916

Eckart C The electrodynamics of material media *Phys. Rev.* **54** 920-923

Eckart C 1960 Variation principles of hydrodynamics *Phys. Fluids* **3** 421

Euler L 1755 Principes généraux du movement des fluides, *Memoire de l'academie des sciences de Berlin* **11** (Berlin Libraires 1757) 274-315; (also) *Leonhardi Euleri, Opera Omnia* ser.2 vol.**XII** ed C Truesdell (Lausanne, Euler Committee of the Swiss Academy of Science) 54-91; [*also* English translation] General laws of the motion of fluids *Fluid Dynamics* **34** (1999) 801-822 https://www.oca.eu/etc7/EE250/historical_note.html

Frankel T 1997 *The Geometry of Physics* (Cambridge: Cambridge University Press) p 654

Herivel J W 1955 The derivation of the equations of motion of an ideal fluid by Hamilton's principle *Proc. Cambridge Phil. Soc.* **51** 344-349

Kambe T 2003 Gauge principle for flows of a perfect fluid *Fluid Dyn. Res.* **32** 193-199

Kambe T 2007a *Elementary Fluid Mechanics* (Singapore: World Scientific) p 386

Kambe T 2007b Gauge principle and variational formulation for ideal fluids with reference to translation symmetry *Fluid Dyn. Res.* **39** 98-120

Kambe T 2008 Variational formulation of ideal fluid flows according to gauge principle *Fluid Dyn. Res.* **40** 399-426.

Kambe T 2010 *Geometrical Theory of Dynamical Systems and Fluid Flows* (Revised ed) Chap.7 (Singapore: World Scientific) p 421

Lamb H 1932 *Hydrodynamics* (Cambridge: Cambridge University Press) p 738

Liepmann H W and Roshko A 1957 Elements of Gasdynamics (New York: John Wiley & Sons) p 443

Lin C C 1963 Hydrodynamics of helium II, *Liquid Helium* Proc. Int. Sch. Phys. *Enrico Fermi* **XXI** (NY: Academic Press) 93-146

Salmon R 1988 Hamiltonian fluid mechanics *Ann. Rev. Fluid Mech.* **20** 225-256

Seliger R L and Whitham G B 1968 Variational principle in continuum mechanics *Proc. Roy. Soc.* A **305** 1-25

Serrin J 1959 Mathematical principles of classical fluid mechanics *Encyclopedia of Physics* (ed S Flügge and C Truesdell: Springer) 125-263

Shapiro A H 1969 Film Notes for Vorticity (National Committee for Fluid Mechanics Films) (Chicago: Encyclopaedia Britannica Edu. Corp.) http://web.mit.edu/hml/ncfmf/09VOR.pdf